MATH 105 101 Midterm 1 Sample 4

1. (15 marks)

(a) (4 marks) Given the function:

$$f(x, y) = \arcsin x^2 + y,$$

find its first order partial derivatives at the point $(0, 3)$. *Simplify your answers.*

Solution: Compute the first order partial derivatives of f:

$$f_x(x, y) = \frac{2x}{\sqrt{1 - x^4}}, \qquad f_y(x, y) = 1.$$

Evaluate those at $(0, 3)$:

$$f_x(0, 3) = 0, \qquad f_y(0, 3) = 1.$$

(b) (3 marks) Find and sketch the domain of the function $f(x, y) = \frac{x}{\sqrt{4x^2+y^2-9}}$.*Label your intercepts.*

Solution: To do division, we need $\sqrt{4x^2 + y^2 - 9} \neq 0$, and to take square root, we need $4x^2 + y^2 - 9 \geq 0$. So, combining these two conditions, we get the domain of f is:

$$D = \{(x, y) \in \mathbb{R}^2 \mid 4x^2 + y^2 > 9\}.$$

The domain of f consists of all points lying strictly outside of the ellipse $4x^2 + y^2 = 9$ centered at $(0, 0)$ with intercepts $(\pm 3/2, 0)$ and $(0, \pm 3)$. The ellipse itself should be dotted since it is not included in the domain.

(c) (3 marks) Let $\mathbf{v} = \langle 3, 2, 1 \rangle$, and $\mathbf{w} = \langle 9, -6, 2 \rangle$. Are the two vectors $\mathbf{v}$ and $\mathbf{w}$ parallel, perpendicular, or neither? Justify your answer.

Solution: If $\mathbf{v}$ is parallel to $\mathbf{w}$, then for some constant c, we get $\mathbf{v} = c\mathbf{w}$, that is:

$$\langle 3, 2, 1 \rangle = \langle 9c, -6c, 2c \rangle.$$

However, $3 = 9c$ implies $c = 1/3$ while $2 = -6c$ implies that $c = -1/3$. So, no such c exists and $\mathbf{v}$ is not parallel to $\mathbf{w}$.

Secondly, we have that:

$$3(9) + 2(-6) + 1(2) = 27 - 12 + 2 = 17 \neq 0,$$

so $\mathbf{v}$ is not perpendicular to $\mathbf{w}$.
Thus, $\mathbf{v}$ is neither parallel nor perpendicular to $\mathbf{w}$.

(d) (2 marks) Can you find a plane parallel to the xy-plane (ie. the plane $z = 0$) passing through the point $P(1, 2, -1)$? If yes, find the equation of this plane. If not, explain why not.

Solution: If $\mathcal{P}$ is parallel to the xy-plane, then it also has the same normal vector as the xy-plane, which is $\mathbf{n} = \langle 0, 0, 1\rangle$. Then, an equation of $\mathcal{P}$ with the normal vector $\mathbf{n}$ passing through the point $P(1, 2, -1)$ is:

$$0(x - 1) + 0(y - 2) + (z + 1) = 0 \Rightarrow z = -1.$$

(e) (3 marks) Is there a function $f(x, y)$ such that $\nabla f(x, y) = \langle \sin y - 1, x \cos y - x\rangle$? If not, explain why no such function exists; otherwise, find $f(x, y)$. Clearly state any result that you may use.

Solution: Suppose that such a function $f(x, y)$ exists. Then, $f_x(x, y) = \sin y - 1$ and $f_y(x, y) = x \cos y - x$. So, $f_{xy}(x, y) = \cos y$ and $f_{yx}(x, y) = \cos y - 1$, which are both continuous on $\mathbb{R}^2$. By Clairaut's Theorem, $f_{xy} = f_{yx}$; however,

$$f_{xy} = \cos y \neq f_{yx} = \cos y - 1,$$

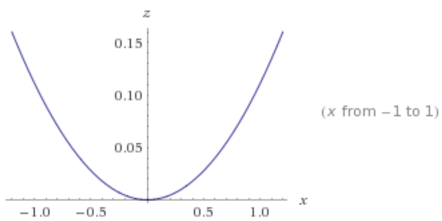
so no such function f exists.

2. (5 marks) Consider the surface S given by:

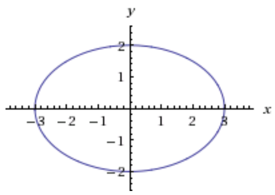
$$z - \frac{x^2}{9} = \frac{y^2}{4}.$$

(a) (4 marks) Find and sketch the traces of S in the $y = 0$ and $z = 1$ planes.

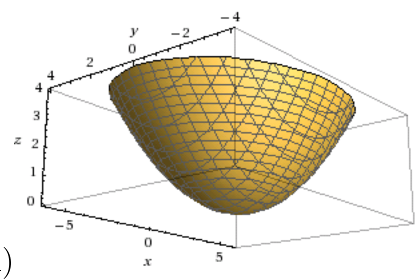
Solution: For $y = 0$, the trace of S is $z = \frac{x^2}{9}$, which is a parabola in the xz-plane.



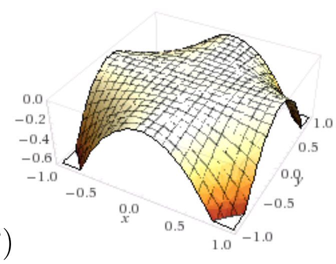
For $z = 1$, the trace of S is $1 = \frac{x^2}{9} + \frac{y^2}{4}$, which is an ellipse centered at $(0, 0)$ with intercepts $(\pm 3, 0)$ and $(\pm 2, 0)$.



(b) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?



(*A*)



(*B*)

Solution: The answer is (A) since in (B), the surface has an empty trace at $z = 1$, not an ellipse.

3. (10 marks) Let R be the semicircular region $\{x^2 + y^2 \leq 9, y \geq x\}$. Find the maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 4x.$$

on the *boundary of the region R*.

Solution: The boundary of the region R consists of two pieces: the semicircular arc which can be parametrized by $x = 3\cos\theta$ and $y = 3\sin\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}$, and the line $y = x$ for $-3/\sqrt{2} \leq x \leq 3/\sqrt{2}$. We will find the potential candidates where the maximum and minimum can occur on each piece:

- *On the semicircular arc*: We have that $f(x, y) = g(\theta) = (3\cos\theta)^2 + (3\sin\theta)^2 - 4(3\cos\theta) = 9 - 12\cos\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{5\pi}{4}$. Then, $g'(\theta) = 12\sin\theta = 0$ if and only if $\theta = \pi$. So, there are 3 points where extrema can occur: $(-3, 0)$ (critical point), $(3/\sqrt{2}, 3/\sqrt{2})$ and $(-3/\sqrt{2}, -3/\sqrt{2})$ (end points).

- *On the slanted line*: We have that $h(x) = x^2 + x^2 - 4x = 2x^2 - 4x$ for $-3/\sqrt{2} \leq x \leq 3/\sqrt{2}$. So, $h'(x) = 4x - 4 = 0$ when $x = 1$. So, there are 3 points where extrema can occur: $(1, 1)$ (critical point), $(3/\sqrt{2}, 3/\sqrt{2})$ and $(-3/\sqrt{2}, -3/\sqrt{2})$ (end points).

Evaluate f at those points, we get:

$f(-3, 0) = 12, \quad f(3/\sqrt{2}, 3\sqrt{2}) = 9 - 12/\sqrt{2}, \quad f(-3/\sqrt{2}, -3/\sqrt{2}) = 9 + 12/\sqrt{2}, \quad f(1, 1) = -2.$

Thus, on the boundary of R, f attains the absolute maximum value $9 + 12/\sqrt{2}$ at the point$(-3/\sqrt{2}, -3/\sqrt{2})$ and the absolute minimum value -3 at the points $(-3, 0)$.

4. (10 marks) Find *all* critical points of the following function:

$$f(x, y) = \frac{x^4}{4} + \frac{y^4}{4} - xy$$

lying in the region $\{(x, y) \mid y \geq 0\}$. Classify each point as a local minimum, local maximum, or saddle point. *You do not have to solve for extrema on the boundary.*

Solution: Compute the first-order partial derivatives of f:

$$f_x(x, y) = x^3 - y \qquad f_y(x, y) \qquad = y^3 - x.$$

Since both f_x and f_y are defined at every point in $\mathbb{R}^2$, the only critical points of f are those at which $f_x = f_y = 0$. If $f_x = 0$, then $y = x^3$. Replacing $y = x^3$ into $f_y = 0$, we get:

$$x^9 - x = x(x^8 - 1) = 0 \Rightarrow x = 0, \pm 1.$$

So, we get three critical points $(0, 0)$, $(-1, -1)$ and $(1, 1)$, among which $(0, 0)$ and $(1, 1)$ lie in the desired region. Compute the second-order partial derivatives and the discriminants,

$$f_{xx} = 3x^2, \quad f_{yy} = 3y^2, \quad f_{xy} = -1, \quad D(x, y) = 9x^2y^2 - 1$$

Using the Second Derivative Test to classify the points, we get:

- At the point $(0, 0)$, $D(0, 0) = -1 < 0$, so $(0, 0)$ is a saddle point.

- At the point $(1, 1)$, $D(1, 1) = 8 > 0$ and $f_{xx}(1, 1) = 3 > 0$, so $(1, 1)$ is a local minimum.

5. (10 marks) A paper company makes two kinds of paper, x units of brown and y units of white every month. Given a fixed amount of raw material, x and y must satisfy the production possibility curve:

$$4x^2 + 25y^2 = 50,000, \text{ for } x \geq 0, y \geq 0.$$

It costs the company \$23 to produce a single unit of brown paper, and \$42 to produce a unit of white. On the other hand, brown and white paper sell for \$25 per unit and \$52 per unit respectively. Assuming that the company is confident of selling all the units it produces, find how many units of brown and white paper it should manufacture every month so as to maximize its total profit, using the method of Lagrange multipliers.

Clearly state the objective function and the constraint. *You are not required to justify that the solution you obtained is the absolute maximum.* **A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.**

Solution: Since it costs the company \$23 to produce a single unit of brown paper, and \$42 to produce a unit of white while brown and white paper sell for \$25 per unit and \$52 per unit respectively, we get the following profit function:

$$P(x, y) = \text{Revenue} - \text{Cost} = (25x + 52y) - (23x + 42y) = 2x + 10y.$$

The objective function to maximize is the profit function $P(x, y) = 2x + 10y$, and the constraint function is $g(x, y) = 4x^2 + 25y^2 - 50000 = 0$. Using Lagrange multiplier, we need to solve the following system of equations:

$$\begin{aligned} \nabla P(x, y) &= \lambda \nabla g(x, y) \\ g(x, y) &= 0 \end{aligned}$$

More explicitly, we need to solve:

$$\begin{aligned} 2 &= \lambda 8x \\ 10 &= \lambda 50y \\ 4x^2 + 25y^2 - 50000 &= 0. \end{aligned}$$

Isolate for λ in the first two equations, we get:

$$\lambda = \frac{1}{4x}, \qquad \lambda = \frac{1}{5y}.$$

Equate the above equations, we get:

$$\frac{1}{4x} = \frac{1}{5y} \Rightarrow 4x = 5y \Rightarrow x = \frac{5}{4}y.$$

Replace $x = \frac{5}{4}y$ in the third equation, we get:

$$\frac{25}{4}y^2 + 25y^2 = 50,000 \Rightarrow y^2 = 1600 \Rightarrow y = 40,$$

since $y \geq 0$. Thus, $y = 40$, $x = 50$ and $\lambda = \frac{1}{200}$. Therefore, the company should manufacture 50 units of brown paper and 40 units of white paper every month to maximize its total profit.